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LETTER TO THE EDITOR

Restricted valence site animals on the triangular lattice

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Abstract. Exact values of the numbers of connected clusters of n sites, each site having valence no larger than v, are presented for the triangular lattice for $v = 2, 3, 4$ and 5 for small values of n. Assuming a plausible asymptotic form for the dependence of these numbers on n and v we show, using series analysis techniques, that the exponent characterising the dominant singularity in the generating function has the same value for all $v \geqslant 3$ but a different value for $v = 2$.

A lattice (site) animal is a cluster of sites of the lattice such that adjacent sites are joined by a bond and every site in the cluster can be reached from every other site in the cluster by a path along these bonds, i.e. a site animal is a connected section graph of the lattice. The numbers of these graphs containing n sites are of interest in the theory of percolation processes (Sykes and Glen 1976) and have been studied by a number of different techniques (e.g. Klarner 1967, Lunnon 1971, Ball and Coxeter 1974, Sykes and Glen 1976, Gaunt *et al* 1976, Whittington and Gaunt 1978). If the number of site animals with n sites, per lattice site, is a_n then Klarner (1967) has shown that

$$0 < \lim_{n\to\infty} n^{-1} \ln a_n \equiv \ln \lambda < \infty \tag{1}$$

and it is reasonable to assume that

$$a_n \sim n^{-\tau} \lambda^n. \tag{2}$$

It seems that the exponent τ is independent of the lattice in a given dimension, but depends on dimension. In two dimensions τ is close to unity so that the singularity in the generating function

$$G(x) = 1 + \sum_{n\geqslant 1} a_n x^n \tag{3}$$

is approximately logarithmic and, close to $x = \lambda^{-1}$,

$$G(x) \sim -A \ln (1 - \lambda x). \tag{4}$$

Recently, Gaunt *et al* (1979) studied the numbers of animals on the square lattice in which the valence of each site in a cluster was not allowed to exceed some pre-assigned number, v, i.e. the number of bonds meeting at a site could not exceed v. If the number of animals with n sites having no vertex of degree greater than v is $a_n(v)$ they showed, for the square lattice, that $a_n(v)^{1/n}$ tends to a limit ($\lambda(v)$, say) for all v and, assuming that

$$a_n(v) \sim n^{-\tau(v)} \lambda(v)^n \tag{5}$$

they showed that $\tau(2)<0$ but $\tau(v)>0$ for $v \geqslant 3$. They also presented numerical evidence that $\tau(3)=\tau(4)=1$. These results are interesting in that (i) they extend the universality class of site animals on two-dimensional lattices and (ii) they show that $v=2$ animals belong to a different universality class. Consequently, it is of interest to consider the behaviour on other lattices, especially one with a coordination number (q) greater than four. This allows an examination of the obvious hypothesis that the exponent changes between $v=2$ and $v=3$ and is then independent of v for larger values of v, for all lattices.

In table 1 we present some exact enumeration data for the triangular lattice. The values for $v=6$ up to $n=16$ are reproduced from Sykes and Glen (1976). For $v=3$, 4 and 5 we have obtained the numbers of clusters for up to 14, 13 and 13 vertices, respectively. For $v=2$, the number of clusters with n vertices is the sum of the number of strongly embedded simple chains with $n-1$ edges, $[n-1]_C$, and the number of strongly embedded polygons with n edges, $[n]_O$,

$$a_n(2)=[n-1]_C+[n]_O. \tag{6}$$

We have enumerated $a_n(2)$ up to $n=17$ and $[n-1]_C$ up to $n=21$. Values of $[n-1]_C$ were previously available only through $n=14$ (Fisher and Hiley 1961, Hioe 1967). As we commented elsewhere (Gaunt *et al* 1979), the dominant asymptotic behaviour of $a_n(2)$ will be the same as $[n-1]_C$, the number of $(n-1)$-step undirected neighbour-avoiding walks (NAW).

Assuming that, close to $x=1/\lambda(v)$,

$$G(x,v)=1+\sum_{n\geqslant 1} a_n(v)x^n \sim [A/(1-\tau)][(1-\lambda x)^{-1+\tau}-1], \tag{7}$$

Table 1. Numbers of clusters $a_n(v)$ with maximum valence v, and simple chains $[n-1]_C$, having n sites and strongly embeddable in the triangular lattice.

n	$[n-1]_C$	$a_n(2)$	$a_n(3)$	$a_n(4)$	$a_n(5)$	$a_n(6)$
1	1	1	1	1	1	1
2	3	3	3	3	3	3
3	9	11	11	11	11	11
4	27	27	44	44	44	44
5	81	81	171	186	186	186
6	237	238	689	808	814	814
7	699	699	2 862	3 585	3 651	3 652
8	2 037	2 040	12 117	16 200	16 677	16 689
9	5 949	5 951	52 002	74 271	77 263	77 359
10	17 277	17 289	225 819	344 460	362 022	362 671
11	50 151	50 169	990 225	1 612 587	1 712 013	1 716 033
12	145 161	145 220	4 377 206	7 608 157	8 158 541	8 182 213
13	419 691	419 811	19 480 313	36 131 209	39 132 064	39 267 086
14	1 211 313	1 211 637	87 198 762			189 492 795
15	3 492 171	3 492 915				918 837 374
16	10 055 403	10 057 308				4 474 080 844
17	28 925 679	28 930 281				
18	83 129 121					
19	238 709 829					
20	684 939 291					
21	1 963 981 569					

we have used standard methods of series analysis (Gaunt and Guttmann 1974) to estimate $\lambda(v)$, $\tau(v)$ and $A(v)$. The results are given in table 2. The estimates for $v=6$ are in excellent agreement with previous work (Sykes and Glen 1976, Guttmann and Gaunt 1978), while our estimates of $\lambda(2)$ and $\tau(2)$ are a considerable improvement over earlier estimates (Fisher and Hiley 1961, Hioe 1967). $\lambda(v)$ appears to be a monotone increasing function of v. $\tau(v)$ appears to have the value unity for all $v \geq 3$, i.e. the singularity in G is logarithmic, while $\tau(2) \simeq -\frac{1}{3}$. This agrees with our results on the square lattice, where $\tau(2) = -\frac{1}{3}$ and $\tau(3) = \tau(4) = 1$. It appears likely that such behaviour, i.e. a change from $-\frac{1}{3}$ to 1 between $v=2$ and 3, will persist for all two-dimensional lattices and that a corresponding change in the sign of the exponent as v changes from 2 to 3 will occur in three dimensions as well. The values obtained for the amplitudes are unremarkable, except to notice that $A(2) \simeq A(q)$, as we also found for the square lattice. The very slow variation of both $A(v)$ and $\lambda(v)$ for the largest values of v should be noted; indeed, to within the quoted uncertainties we have $A(4) = A(5) = A(6)$ and $\lambda(5) = \lambda(6)$, although we are not necessarily speculating that these equalities hold exactly.

Table 2. Estimates of critical parameters for triangular lattice.

v	$\lambda(v)$	$\tau(v)$	$A(v)$
2	$2\cdot826 \pm 0\cdot004$	$-\frac{1}{3} \pm 0\cdot06$	$0\cdot272 \pm 0\cdot010$
3	$4\cdot815 \pm 0\cdot005$	$1 \pm 0\cdot02$	$0\cdot343 \pm 0\cdot008$
4	$5\cdot135 \pm 0\cdot010$	$1 \pm 0\cdot03$	$0\cdot280 \pm 0\cdot008$
5	$5\cdot180 \pm 0\cdot010$	$1 \pm 0\cdot03$	$0\cdot276 \pm 0\cdot008$
6	$5\cdot183 \pm 0\cdot007$	$1 \pm 0\cdot02$	$0\cdot274 \pm 0\cdot006$

Using Padé approximant techniques we have examined each of the series to try to identify the subdominant singularities. For $v \geq 3$ the singularity at $x = 1/\lambda(v)$ is the only one of any strength within a circle of radius about $3/\lambda(v)$ centred on the origin. For $v=2$ there appears to be an additional singularity, located on the negative real axis. Although its precise position is rather difficult to ascertain, it probably lies at $x = -1/\lambda(2)$ as is also found (unpublished work) for the generating function of NAW. The generating function of self-avoiding walks (SAW) has a singularity at $x = 1/\mu$ (where the SAW limit μ is the analogue of λ) but not at $-1/\mu$ (Hioe 1967, Watts 1975). This situation is analogous to the absence of an antiferromagnetic singularity in the high-temperature expansion of the zero-field susceptibility for the triangular lattice and so would not be expected to obtain for loose-packed lattices. For the square (Gaunt *et al* 1979) and honeycomb lattices, we have confirmed that symmetrically placed singularities occur in all three generating functions.

We have also added two terms to the $v=2$ and $v=3$ series for the square lattice:

$$a_{20}(2) = 14\,319\,334 \qquad a_{21}(2) = 33\,687\,146 \tag{8}$$

and

$$a_{15}(3) = 20\,365\,888 \qquad a_{16}(3) = 75\,559\,395. \tag{9}$$

The earlier terms are tabulated by Gaunt *et al* (1979). Reanalysing the series including these extra terms simply confirms our previous estimates of λ, τ and A. For completeness we also give here the number of (undirected) NAW on the square lattice with up to

$n = 25$ sites, which form the dominant contribution to $a_n(2)$. The generating function is

$$
\begin{aligned}
N(x) &= 1 + \sum_{n\geq 1} [n-1]_C x^n \\
&= 1 + 1x + 2x^2 + 6x^3 + 14x^4 + 34x^5 + 82x^6 + 198x^7 + 470x^8 + 1\,122x^9 \\
&\quad + 2\,662x^{10} + 6\,334x^{11} + 14\,970x^{12} + 35\,506x^{13} + 83\,734x^{14} \\
&\quad + 198\,086x^{15} + 466\,314x^{16} + 1\,100\,818x^{17} + 2\,587\,634x^{18} \\
&\quad + 6\,097\,830x^{19} + 14\,316\,402x^{20} + 33\,687\,146x^{21} + 79\,008\,870x^{22} \\
&\quad + 185\,677\,006x^{23} + 435\,098\,774x^{24} + 1\,021\,404\,998x^{25} + \ldots
\end{aligned}
\tag{10}
$$

and was previously available (Fisher and Hiley 1961, Hioe 1967; also Domb, Gillis and Wilmers, unpublished) only through order x^{19}.

On the honeycomb lattice we have enumerated undirected NAW with up to $n = 35$ sites giving

$$
\begin{aligned}
N(x) &= 1 + 1x + 1\tfrac{1}{2}x^2 + 3x^3 + 6x^4 + 12x^5 + 21x^6 + 39x^7 + 72x^8 + 132x^9 + 243x^{10} \\
&\quad + 447x^{11} + 810x^{12} + 1\,482x^{13} + 2\,688x^{14} + 4\,899x^{15} + 8\,880x^{16} \\
&\quad + 16\,146x^{17} + 29\,199x^{18} + 52\,980x^{19} + 95\,733x^{20} + 173\,427x^{21} \\
&\quad + 313\,086x^{22} + 566\,400x^{23} + 1\,021\,623x^{24} + 1\,846\,203x^{25} \\
&\quad + 3\,327\,534x^{26} + 6\,007\,563x^{27} + 10\,820\,763x^{28} + 19\,519\,905x^{29} \\
&\quad + 35\,138\,508x^{30} + 63\,341\,292x^{31} + 113\,964\,390x^{32} \\
&\quad + 205\,302\,499x^{33} + 369\,211\,746x^{34} + 664\,738\,866x^{35} + \ldots .
\end{aligned}
\tag{11}
$$

We do not know of any earlier data for this problem. Series analysis of (11) suggests $\lambda(2) = 1{\cdot}7832 \pm 0{\cdot}0008$, $\tau(2) = -\frac{1}{3} \pm 0{\cdot}015$ and $A(2) = 0{\cdot}381 \pm 0{\cdot}010$. The values of $a_n(3)$ up to $n = 22$ are tabulated by Sykes and Glen (1976), who estimate $\lambda(3) = 3{\cdot}04 \pm 0{\cdot}02$ and that $\tau(3)$ is 'very close to unity'. We have used their series to estimate $A(3) = 0{\cdot}382 \pm 0{\cdot}001 - 2{\cdot}5\Delta\lambda$ where $\Delta\lambda$ represents a change in $\lambda(3)$. Once again we have $A(2) \simeq A(q)$.

To summarise, we have found that for site clusters on the triangular, square and honeycomb lattices the exponent τ changes between $v = 2$ and $v = 3$ from $\tau(2) \simeq -\frac{1}{3}$ to $\tau(3) \simeq 1$ and is then independent of v for all larger values of v.

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